

Multiquadric B-splines

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Our purpose in this paper is to show that the analogy between polynomial splines and generalized multiquadrics is very strong. In particular, combinations of multiquadrics, called ψ -splines are defined that are analogues of polynomial B-splines. This paper includes global linear independence, polynomial reproduction, and quasi-interpolation results for the span of the ψ -splines on non-uniform bi-infinite meshes which parallel those for polynomial B-splines. There are also results concerning the relationship between certain semi-infinite and bi-infinite combinations of ψ -splines. These results enable us to obtain error estimates for quasi-interpolation schemes involving multiquadrics based on a finite number of centres. © 1996 Academic Press, Inc.

1. INTRODUCTION

There has recently been a great deal of interest in approximation by radial basis functions, that is, by sums of translates of a single radially symmetric function. From this point of view univariate polynomial splines of odd degree are formed from polynomials plus sums of translates of the modulus raised to a fixed odd power. The generalized multiquadrics can then be viewed as obtained by smoothing out the derivative discontinuity of the function $|\cdot|^{2k-1}$. More precisely, let $k \in \mathbb{N}$ and $c > 0$. Then the basic generalized multiquadric of order $2k$ is defined by

$$\phi(x; 2k) = (x^2 + c^2)^{(2k-1)/2}, \quad (1.1)$$

and the ϕ function centered at t_j by

$$\phi_{j, 2k}(x) = \phi(x - t_j; 2k). \quad (1.2)$$

A multiquadric spline based on a finite number of centers is then a linear combination of appropriate $\phi_{j, 2k}$'s supplemented by a polynomial of degree $2k - 1$.

Our purpose in this paper is to show that the analogy between polynomial splines and generalized multiquadrics is very strong. In particular combinations of multiquadrics, called ψ -splines, will be defined that are analogues of polynomial B-splines. The paper includes global linear independence, polynomial reproduction, and quasi-interpolation results for the span of the ψ -splines on non-uniform bi-infinite meshes which parallel those for polynomial B-splines. Furthermore, it is shown that if a polynomial is expressed as a bi-infinite series of ψ -splines then corresponding semi-infinite series sum to half the polynomial plus a few generalized multiquadrics. This result allows us to obtain error estimates for quasi-interpolation by generalized multiquadrics based on a finite number of centers, from the results for quasi-interpolation by ψ -splines on non-uniform bi-infinite meshes. Our results extend those of Powell [7] and Beatson and Powell [2] for the case $k = 1$ to general $k \in \mathbb{N}$.

2. PRELIMINARIES

In this section we define the ψ -splines and obtain some identities involving them.

Consider a mesh $\mathbf{t} = \dots < t_{j-1} < t_j < t_{j+1} < \dots$, with $t_{\pm j} \rightarrow \pm \infty$ as $j \rightarrow \infty$. Define the ψ -spline $\psi_{j, 2k} (= \psi_{j, 2k, t})$ as the weighted divided difference

$$\psi_{j, 2k}(x) = \frac{t_{j+2k} - t_j}{2} [x - t_j, x - t_{j+1}, \dots, x - t_{j+2k}] \phi(x; 2k). \quad (2.1)$$

Hence making use of the expression of a divided difference as a linear combination of the function values at the indicated points, we can rewrite (2.1) as

$$\psi_{j, 2k}(x) = \frac{t_{j+2k} - t_j}{2} [t_j, t_{j+1}, \dots, t_{j+2k}]_u \phi(x - u; 2k), \quad (2.2)$$

where the subscript u , which will often be omitted, indicates that the divided difference is taken with respect to the u variable. This combination

of $\phi_{j, 2k}, \dots, \phi_{j+2k, 2k}$ will turn out to have some critical properties in common with the B-spline $N_{j, 2k, t}$. (As is usual $N_{j, 2k, t}$ denotes the B-spline of order $2k$ supported on $[t_j, t_{j+2k}]$ and normalized so that the sum of all the B-splines of a fixed order is 1.)

Before proceeding we need to derive some properties of the functions $\phi(x; 2k)$ and their derivatives and integrals. Firstly

$$D^{2k} \phi(x; 2k) = [(2k-1)!!]^2 \frac{c^{2k}}{(x^2 + c^2)^{(2k+1)/2}}, \quad k \in \mathbf{N}, \quad (2.3)$$

where as usual

$$m!! = \prod_{\{j: j \equiv m \pmod{2} \text{ and } 0 < j \leq m\}} j.$$

Equation (2.3) can be shown by induction, the induction step following from applying Leibnitz's rule to

$$D^{2k+2} \{ (x^2 + c^2)(x^2 + c^2)^{(2k-1)/2} \}.$$

Proceeding from (2.3) a similar induction argument shows that

$$D^{2k-1} \phi(x; 2k) = \frac{p(x; 2k)}{(x^2 + c^2)^{(2k-1)/2}}, \quad k \in \mathbf{N} \quad (2.4)$$

where $p(x; 2k)$ is an odd polynomial in x defined by the recurrence

$$p(x; 2k+2) = \begin{cases} x, & k=0, \\ (2k+1)[(2k-1)!!]^2 c^{2k} x \\ \quad + (2k+1) 2k(x^2 + c^2) p(x; 2k), & k \in \mathbf{N}. \end{cases} \quad (2.5)$$

It follows immediately from this recurrence that the power expansion of $p(x; 2k)$ has positive coefficients and that

$$p(x; 2k) = (2k-1)! x^{2k-1} + \sum_{j=1}^{k-1} a_{j, 2k} c^{2k-2j} x^{2j-1}, \quad (2.6)$$

for some constants $a_{j, 2k}$ not depending on c . Hence

$$D^{2k-1} \phi(x; 2k) = \pm (2k-1)! + \mathcal{O}(|x|^{-2}), \quad \text{as } x \rightarrow \pm \infty. \quad (2.7)$$

Defining

$$A(k) = \frac{(2k-1)!!}{(2k)!!} = \frac{1}{2} \frac{3}{4} \frac{5}{6} \dots \frac{(2k-1)}{2k} \quad (2.8)$$

we have that

$$\int_{-\infty}^{\infty} (x^2 + c^2)^{-(2k+1)/2} dx = \frac{1}{kA(k)c^{2k}}, \quad k \in \mathbf{N}. \quad (2.9)$$

This can be proven by induction, the induction step following from the easily verified identity

$$\begin{aligned} \int (x^2 + c^2)^{-(2k+1)/2} dx &= \frac{x}{(2k-1)c^2} (x^2 + c^2)^{-(2k-1)/2} \\ &\quad + \frac{2(k-1)}{(2k-1)c^2} \int (x^2 + c^2)^{-(2k-1)/2} dx. \end{aligned} \quad (2.10)$$

Also trivially

$$\int x(x^2 + c^2)^{-(2k+1)/2} dx = -\frac{1}{2k-1} (x^2 + c^2)^{-(2k-1)/2} + M, \quad (2.11)$$

so that

$$\int_{-\infty}^{\infty} |x| (x^2 + c^2)^{-(2k+1)/2} dx = \frac{2}{(2k-1)c^{2k-1}}. \quad (2.12)$$

Recall now that a multiple of the polynomial B-spline is the Peano kernel of the divided difference, so that in particular

$$[t_j, \dots, t_{j+2k}] g = \frac{2k}{(t_{j+2k} - t_j)} \frac{1}{(2k)!} \int_{-\infty}^{\infty} N_{j, 2k, t}(u) g^{(2k)}(u) du. \quad (2.13)$$

Using this with $g(u) = \phi(x - u; 2k)$ we find that

$$\psi_{j, 2k}(x) = \frac{k}{(2k)!} \int_{-\infty}^{\infty} N_{j, 2k, t}(u) \phi^{(2k)}(x - u) du. \quad (2.14)$$

Applying (2.3)

$$\psi_{j, 2k}(x) = kA(k)c^{2k} \int_{-\infty}^{\infty} N_{j, 2k, t}(u) ((x - u)^2 + c^2)^{-(2k+1)/2} du. \quad (2.15)$$

It follows from this and the formula

$$\int_{-\infty}^{\infty} N_{j, l, t}(x) dx = \frac{t_{j+l} - t_j}{l}, \quad (2.16)$$

for the integral of a B-spline that $\psi_{j, 2k}$ is nonnegative and decays like $[t_{j+2k} - t_j] d(x, [t_j, t_{j+2k}])^{-(2k+1)}$ as $x \rightarrow \pm \infty$, where $d(\cdot, \cdot)$ denotes the usual distance for \mathbf{R} . Also

$$\begin{aligned} \sum_{j=-\infty}^{\infty} \psi_{j, 2k}(x) &= kA(k) c^{2k} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} N_{j, 2k, t}(u) ((x-u)^2 + c^2)^{-(2k+1)/2} du, \\ &= kA(k) c^{2k} \int_{-\infty}^{\infty} ((x-u)^2 + c^2)^{-(2k+1)/2} du, \\ &= 1, \end{aligned} \tag{2.17}$$

where in the last step we have used (2.9). Let

$$S_{2k}(u) = kA(k) c^{2k} (u^2 + c^2)^{-(2k+1)/2}, \quad k \in N.$$

Jones [6, p. 178] gives the formula

$$\int_{-\infty}^{\infty} e^{-itx} (1+x^2)^{-(k+1/2)} dx = \frac{\pi^{1/2} |t|^k K_k(|t|)}{(k - \frac{1}{2})! 2^{k-1}},$$

where K_k is a modified Bessel function. Hence the Fourier transform of S_{2k} is

$$\widehat{S_{2k}}(t) = kA(k) c^{2k} \frac{2\pi^{1/2}}{(k - \frac{1}{2})!} \left| \frac{t}{2c} \right|^k K_k(|ct|). \tag{2.18}$$

Now from Abromowitz and Stegun [1, p. 375]

$$\begin{aligned} K_k(z) &= \frac{1}{2} \left(\frac{1}{2} z \right)^{-k} \frac{(k-j-1)!}{j!} \left(-\frac{1}{4} z^2 \right)^j \\ &\quad + (-)^{k+1} \ln \left(\frac{1}{2} z \right) I_k(z) \\ &\quad + (-)^k \frac{1}{2} \left(\frac{1}{2} z \right)^k \sum_{j=0}^{\infty} (\eta(j+1) + \eta(k+j+1)) \frac{(\frac{1}{4} z^2)^j}{j!(k+j)!} \end{aligned}$$

where here η denotes the digamma function, and

$$I_k(z) = \frac{(\frac{1}{2}z)^k}{\Gamma(k+1)} + \mathcal{O}(z^{k+2}), \quad \text{as } z \rightarrow 0.$$

Hence

$$\begin{aligned} \widehat{S}_{2k}(t) &= \sum_{j=0}^{k-1} \frac{(k-j-1)!}{(k-1)! j!} \left(\frac{-c^2}{4} \right)^4 t^{2j} \\ &\quad + kA(k)c^{2k} \left\{ (-)^{k+1} \frac{2\pi^{1/2}}{(k-\frac{1}{2})!} \left(\frac{t}{2} \right)^{2k} \ln \left(\frac{c|t|}{2} \right) (1 + \mathcal{O}(c^2 t^2)) \right. \\ &\quad \left. + \mathcal{O}(t^{2k}) \right\} \quad \text{as } t \rightarrow 0. \end{aligned} \quad (2.19)$$

Since $K_k(z)$ is infinitely differentiable on $\mathbf{R} \setminus \{0\}$, it follows from (2.19) that $\widehat{S}_{2k} \in C^{2k-1}(\mathbf{R})$. Since $K_k(z)$ is positive for $z > 0$ and $k > -1$, $\widehat{S}_{2k}(t)$ is positive for $t \in \mathbf{R}$. Also from (2.19)

$$\widehat{S}_{2k}^{(2j)}(0) = \frac{(k-j-1)!}{(k-1)! j!} \left(\frac{-c^2}{4} \right)^j (2j)!, \quad 0 \leq j \leq k-1.$$

Hence from the formula $(\widehat{x^r f})(t) = (id/dt)^r \widehat{f}(t)$

$$\int_{\mathbf{R}} x^{2j} S_{2k}(x) dx = \frac{(2j)! (k-j-1)!}{(k-1)! j!} \left(\frac{c^2}{4} \right)^j \quad (2.20)$$

for $0 \leq j \leq k-1$. Applying Cauchy–Schwarz we have

$$\begin{aligned} \int_{\mathbf{R}} |x|^{2j-1} S_{2k}(x) dx &\leq \left[\int_{\mathbf{R}} x^{2j} S_{2k}(x) dx \int_{\mathbf{R}} x^{2j-2} S_{2k}(x) dx \right]^{1/2} \\ &= \mathcal{O}(c^{2j-1}), \quad 0 \leq j \leq k-1. \end{aligned}$$

Also note that in the particular case $j=1$ (2.20) gives

$$\int_{\mathbf{R}} x^2 S_{2k}(x) dx = \frac{c^2}{2(k-1)}, \quad k \geq 2. \quad (2.21)$$

3. BASIC PROPERTIES OF ψ -SPLINES

In this section we derive some fundamental properties of the ψ -splines. These include global linear independence, polynomial reproduction properties, and expressions for ϕ and ψ splines as convolutions of a kernel with a power of the modulus and polynomial B-splines respectively.

It will be convenient to have the following notation. Given an infinite mesh $\mathbf{t} : \cdots < t_{j-1} < t_j < t_{j+1} < \cdots$ we define a coefficient sequence $\mathbf{d} = \{d_j\}_{j=-\infty}^{\infty}$ to be in the growth class $C(2k, \mathbf{t})$ if $d_j = \mathcal{O}(|t_j|^{2k-1})$ as $j \rightarrow \pm\infty$. We note

that for meshes \mathbf{t} of finite mesh size the condition is equivalent to the condition $\sum_{j=-\infty}^{\infty} d_j N_{j,2k}(x) = \mathcal{O}(|x|^{2k-1})$ as $x \rightarrow \pm\infty$. (See the proof of Lemma 3.)

We remind the reader of the following well known result.

LEMMA 1 (Local linear independence of the B-spline basis). *Let $k \in \mathbf{N}$ and consider an infinite mesh $\mathbf{t}: \dots < t_{j-1} < t_j < t_{j+1} < \dots$. Let $s = \sum_{j=-\infty}^{\infty} a_j N_{j,k}$ and $r = \sum_{j=-\infty}^{\infty} b_j N_{j,k}$. Then $s(x) = r(x)$ for all $x \in (t_j, t_{j+1})$ if and only if $a_l = b_l$ for all $j-k < l \leq j$.*

Proof. Let p be any polynomial of degree $k-1$. Then p can be expressed as a linear combination of B-splines of order k . Because of the support properties of the B-splines this implies that $E := \{N_{l,k} : j-k < l \leq j\}$ is a spanning set for the polynomials of degree $k-1$ considered as a vector space of functions from (t_j, t_{j+1}) to \mathbf{R} . From the cardinality of E it follows that E is not merely a spanning set but also a basis for this vector space. This is the required result. ■

The next result expresses ψ -splines as a convolution. It is a generalization of a result of Powell [7] in the case of the ordinary multiquadric.

LEMMA 2 (ψ -splines as convolutions). *Let $k \in \mathbf{N}$ and consider an infinite mesh*

$$\mathbf{t}: \dots < t_{j-1} < t_j < t_{j+1} < \dots, \quad t_{\pm j} \rightarrow \pm\infty, \quad \text{as } j \rightarrow \infty,$$

and $h = \sup_j (t_{j+1} - t_j) < \infty$. Suppose $\alpha = \{\alpha_j\}_{j=-\infty}^{\infty} \in C(2k, \mathbf{t})$. Then

$$s(x) = \sum_{j=-\infty}^{\infty} \alpha_j \psi_{j,2k,t}(x)$$

is absolutely convergent for each x and is given by

$$s(x) = kA(k)c^{2k} \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \alpha_j N_{j,2k,t}(u) ((x-u)^2 + c^2)^{-(2k+1)/2} du$$

in which the integral is absolutely convergent.

Proof. Define g as the B-spline series

$$g(x) = \sum_j \alpha_j N_{j,2k,t}(x)$$

and

$$M(x) = \sum_j |\alpha_j| N_{j, 2k, t}(x).$$

As is familiar there is no convergence problem with these series as only a finite number of the B -splines are non-zero at any x . Indeed on $[t_i, t_{i+1}]$ only $N_{i-2k+1, 2k, t}, \dots, N_{i, 2k, t}$ are non-zero. From this, the growth condition on α , the partition of unity property of the B -splines, and the finiteness of the mesh size, it follows that

$$|g(x)| \leq M(x) = \mathcal{O}(|x|^{2k-1}) \quad \text{as } x \rightarrow \pm\infty.$$

Hence, for each fixed, x

$$0 \leq kA(k) c^{2k} M(u)((x-u)^2 + c^2)^{-(2k+1)/2} = \mathcal{O}(u^{-2}), \quad \text{as } u \rightarrow \pm\infty.$$

Thus the middle quantity above is integrable with respect to u on \mathbf{R} . It now follows from (2.15) and the Lebesgue dominated convergence theorem that

$$\sum_{j=-\infty}^{\infty} \alpha_j \psi_{j, 2k, t}(x) = kA(k) c^{2k} \int_{-\infty}^{\infty} \sum_j \alpha_j N_{j, 2k, t}(u)((x-u)^2 + c^2)^{-(2k+1)/2} du$$

with the series on the left converging absolutely for each x . ■

LEMMA 3. (Polynomials in the space spanned by the ψ -splines). *Let \mathbf{t} satisfy the conditions of Lemma 2. Suppose that $p \in \pi_{2k-1}$ has B -spline series expansion*

$$p(x) = \sum_j d_j N_{j, 2k}(x).$$

Then

$$s(x) = \sum_j d_j \psi_{j, 2k}(x),$$

is a polynomial of the same degree as p and with the same leading coefficient. Moreover, if $p \in \pi_1$ then s and p are identical and

$$d_j = p(t_j^*), \quad j = 0, \pm 1, \pm 2, \dots,$$

where the points

$$t_j^* = \frac{t_{j+1} + \dots + t_{j+2k-1}}{2k-1},$$

are the special points occurring in the definition of Schoenberg's variation diminishing spline.

Proof. Recall the remarkable condition property of the B -spline basis (see for example de Boor [5, p. 155])

$$|d_i| \leq D_{2k} \left\| \sum_j d_j N_{j, 2k} \right\|_{L^\infty[t_{i+1}, t_{i+2k-1}]}$$

where D_{2k} is independent of the mesh. Hence

$$|d_i| \leq D_{2k} \|p\|_{L^\infty[t_{i+1}, t_{i+2k-1}]}.$$

This together with the finiteness of the mesh size, implies that the coefficients belong to the growth class $C(2k, \mathbf{t})$. Hence, from Lemma 2,

$$\begin{aligned} s(x) &= kA(k)c^{2k} \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} d_j N_{j, 2k}(u) ((x-u)^2 + c^2)^{-(2k+1)/2} du \\ &= kA(k)c^{2k} \int_{-\infty}^{\infty} p(x-u)(u^2 + c^2)^{-(2k+1)/2} du. \end{aligned} \quad (3.1)$$

Supposing now p is of exact degree m , $0 \leq m \leq 2k-1$, so that

$$p(t) = a_m t^m + a_{m-1} t^{m-1} + \dots + a_0,$$

with $a_m \neq 0$, (3.1) implies

$$\begin{aligned} s(x) &= kA(k)c^{2k} a_m x^m \int_{-\infty}^{\infty} (u^2 + c^2)^{-(2k+1)/2} du \\ &\quad + \sum_{i=0}^{m-1} b_i x^i \int_{-\infty}^{\infty} p_i(u)(u^2 + c^2)^{-(2k+1)/2} du, \end{aligned}$$

where $p_i(u) \in \pi_m$. The first part of the lemma follows.

For $p \in \pi_1$ Schoenberg's variation diminishing spline with coefficients $d_j = p(t_j^*)$ satisfies

$$p(x) = \sum_j p(t_j^*) N_{j, 2k}(x), \quad \forall x.$$

From Lemma 1, that is the local linear independence of the B-splines, the coefficients $p(t_j^*)$ are the only coefficients with this property. An application of (3.1) now gives

$$\begin{aligned}
s(x) &= \sum_j p(t_j^*) \psi_{j, 2k}(x) \\
&= kA(k) c^{2k} \int_{-\infty}^{\infty} p(x-u)(u^2 + c^2)^{-(2k+1)/2} du \\
&= kA(k) c^{2k} p(x) \int_{-\infty}^{\infty} (u^2 + c^2)^{-(2k+1)/2} du \\
&= p(x),
\end{aligned}$$

where in the second to last step we have used that if g is odd and integrable $\int_{-\infty}^{\infty} g = 0$. ■

Unfortunately when p has degree greater than 1 the coefficients used to express p in terms of the B -splines do not suffice to express it in terms of the ψ -splines. For example if $k > 1$ and

$$p(x) = x^2 = \sum_j d_j N_{j, 2k}(x),$$

then by (2.21)

$$\begin{aligned}
s(x) &= \sum_j d_j \psi_{j, 2k}(x) \\
&= x^2 + kA(k) c^{2k} \int_{-\infty}^{\infty} u^2 (u^2 + c^2)^{-(2k+1)/2} du \\
&= x^2 + Mc^2
\end{aligned}$$

where the non-zero constant M , depends on k .

LEMMA 4 (Global linear independence of the ψ -spline basis). *Let \mathbf{t} be as in Lemma 2 and suppose $\mathbf{d} \in C(2k, \mathbf{t})$. Then*

$$s(x) = \sum_j d_j \psi_{j, 2k}(x) = 0, \quad \text{for all } x,$$

implies \mathbf{d} is the zero sequence.

Proof. The assumptions on \mathbf{t} and \mathbf{d} ensure

$$g = \sum_j d_j N_{j, 2k},$$

satisfies $g(x) = \mathcal{O}(|x|^{2k-1})$ as $x \rightarrow \pm\infty$ hence is a tempered distribution, having a generalized Fourier Transform well defined except possibly at

zero. (The properties of tempered distributions we use can be found in Rudin [8, particularly pp. 173–178].) Now

$$0 = s(x) = \sum_j d_j \psi_{j, 2k}(x), \quad \text{for all } x,$$

implies by Lemma 2

$$0 = kA(k)c^{2k} \left\{ \sum_j d_j N_{j, 2k}(\cdot) \right\} * (\cdot^2 + c^2)^{-(2k+1/2)}, \quad \text{for all } x. \quad (3.2)$$

Taking the generalized Fourier transform we obtain

$$0 = \hat{g}(\xi) \widehat{S}_{2k}(\xi), \quad \text{for all } \xi \neq 0, \quad (3.3)$$

where $\widehat{S}_{2k} \in C^{2k-1}(\mathbf{R})$ is the everywhere positive transform of S_{2k} , previously discussed (see (2.18) and (2.19)). Hence (3.3) implies the support of $\hat{g}(\xi)$ is $\xi = 0$. Thus g must be a polynomial. Since from above $g(x) = O(|x|^{2k-1})$, this polynomial has exact degree not exceeding $2k-1$. Then, from Lemma 3, s is a polynomial of the same exact degree. But from the hypotheses s being identically zero has exact degree -1 . Hence $\sum d_j N_{j, 2k}$ is identically zero. The result follows from Lemma 1. ■

The following corollary goes in the opposite direction to Lemma 3.

COROLLARY 5 (Polynomial reproduction and a dual representation). *Let $k \in \mathbf{N}$, and \mathbf{t} be as in Lemma 2. Let*

$$s(x) = \sum_j d_j \psi_{j, 2k}(x) \quad \text{and} \quad g(x) = \sum_j d_j N_{j, 2k}(x). \quad (3.4)$$

where the first sum may be divergent. Then:

- (a) *Given any polynomial $q \in \pi_{2k-1}$ there is a unique choice of coefficients \mathbf{d} such that both $s = q$ and the growth condition $\mathbf{d} \in C(2k, \mathbf{t})$ hold.*
- (b) *If $\mathbf{d} \in C(2k, \mathbf{t})$ and $s \in \pi_{2k-1}$ then $g \in \pi_{2k-1}$ and has the same exact degree and leading coefficient as s .*
- (c) *If $\mathbf{d} \in C(2k, \mathbf{t})$ and $s \in \pi_1$ then s and g are identical.*

Proof. (a) Let $q \in \pi_{2k-1}$ be fixed and of exact degree m . From Lemma 3 choosing $p(x)$ there as $x^m, x^{m-1}, \dots, 1$ in turn, and then taking linear combinations, we can find coefficients \mathbf{d} satisfying the growth condition $\mathbf{d} \in C(2k, \mathbf{t})$ and $s = q$. From Lemma 4 these coefficients are unique. Note that in this construction $\sum_j d_j N_{j, 2k}$ is a polynomial with the same exact degree and leading coefficient as q .

(b) Let $\mathbf{d} \in C(2k, \mathfrak{t})$ and $s \in \pi_{2k-1}$. Then from the uniqueness part of part (a) the coefficients \mathbf{d} must be those of the construction of part (a). The conclusion follows from the remark at the end of the proof of part (a).

(c) From (b) if $s \in \pi_1$ then $g \in \pi_1$. The conclusion then follows from Lemma 3.

LEMMA 6. *Let $k \in \mathbf{N}$ and $c > 0$. Then*

$$I_k = \int \frac{u^{2k-1}}{((x-u)^2 + c^2)^{(2k+1)/2}} du = \left\{ \frac{p(x,u)}{((x-u)^2 + c^2)^{(2k-1)/2}} \right\} + C, \quad (3.5)$$

where $p(x, u)$, considered as a polynomial in u , has degree $2k-1$ and constant part

$$-\frac{(x^2 + c^2)^{2k-1}}{2kA(k)c^{2k}}. \quad (3.6)$$

Proof. By differentiation one easily establishes the recurrences

$$\begin{aligned} & \int \frac{u^m}{((x-u)^2 + c^2)^{(2k+1)/2}} du \\ &= \frac{u^{m-1}}{(m-2k)((x-u)^2 + c^2)^{(2k-1)/2}} \\ & \quad - \left(\frac{2k-2m+1}{m-2k} \right) x \int \frac{u^{m-1}}{((x-u)^2 + c^2)^{(2k+1)/2}} du \\ & \quad - \left(\frac{m-1}{m-2k} \right) (x^2 + c^2) \int \frac{u^{m-2}}{((x-u)^2 + c^2)^{(2k+1)/2}} du \end{aligned}$$

and

$$\begin{aligned} \int ((x-u)^2 + c^2)^{-(2k+1)/2} du &= \frac{(u-x)}{(2k-1)c^2} ((x-u)^2 + c^2)^{-(2k-1)/2} \\ & \quad + \frac{2(k-1)}{(2k-1)c^2} \int ((x-u)^2 + c^2)^{-(2k-1)/2} du. \end{aligned}$$

Since

$$\int ((x-u)^2 + c^2)^{-3/2} du = \frac{(u-x)}{c^2} ((x-u)^2 + c^2)^{-1/2} + D,$$

an easy induction shows that there is an indefinite integral I_k of the form stated in (3.5). It remains to show that the constant part of the polynomial in u , $p(x, u)$, is given by (3.6).

To this end make the substitution

$$\cos \theta = \frac{c}{\sqrt{(x-u)^2 + c^2}} \quad \text{and} \quad \sin \theta = \frac{-(x-u)}{\sqrt{(x-u)^2 + c^2}},$$

implying $u = x + c \tan \theta$. Note in particular that the expression defining $\cos \theta$ is defined everywhere and is always positive. Then

$$\begin{aligned} I_k &= \int \frac{(x + c \tan \theta)^{2k-1} c \sec^2 \theta}{(c \sec \theta)^{2k+1}} d\theta \\ &= \frac{1}{c^{2k}} \int (x \cos \theta + c \sin \theta)^{2k-1} d\theta \\ &= \frac{(x^2 + c^2)^{(2k-1)/2}}{c^{2k}} \int \cos^{2k-1} t dt \end{aligned}$$

where

$$t = \theta - \gamma, \quad \cos \gamma = \frac{x}{\sqrt{x^2 + c^2}} \quad \text{and} \quad \sin \gamma = \frac{c}{\sqrt{x^2 + c^2}}.$$

Then with $v = \sin t$

$$\int \cos^{2k-1} t dt = \int (1 - v^2)^{k-1} dv = \sum_{j=0}^{k-1} \frac{\binom{k-1}{j} (-1)^j v^{2j+1}}{2j+1} + E.$$

Therefore we may choose

$$p(x, u) = \left(\frac{c}{\cos \theta} \right)^{2k-1} \frac{(x^2 + c^2)^{(2k-1)/2}}{c^{2k}} \sum_{j=0}^{k-1} \frac{\binom{k-1}{j} (-1)^j \sin^{2j+1} t}{2j+1}.$$

When $u = 0$, $\cos \theta = c/\sqrt{x^2 + c^2} = \sin \gamma$, $\sin \theta = -x/\sqrt{x^2 + c^2} = -\cos \gamma$, and $\sin t = \sin(\theta - \gamma) = \cos \gamma \sin \theta - \cos \theta \sin \gamma = -1$. Therefore

$$p(x, 0) = -\frac{(x^2 + c^2)^{2k-1}}{c^{2k}} \sum_{j=0}^{k-1} \frac{(-1)^j \binom{k-1}{j}}{2j+1}.$$

Finally

$$\frac{1}{2kA(k)} = \frac{(2k-2)!!}{(2k-1)!!} = \int_0^{\pi/2} \sin^{2k-1} x \, dx = \sum_{j=0}^{k-1} \frac{(-1)^j \binom{k-1}{j}}{2j+1},$$

which completes the proof. ■

LEMMA 7 (ϕ and ψ splines as convolutions). *Let $k \in \mathbf{N}$. Then*

$$\phi(x; 2k) = kA(k) c^{2k} |\cdot|^{2k-1} * (\cdot^2 + c^2)^{-(2k+1)/2}, \quad (3.7)$$

and

$$\psi_{j, 2k}(x) = kA(k) c^{2k} N_{j, 2k, t} * (\cdot^2 + c^2)^{-(2k+1)/2}. \quad (3.8)$$

Proof. For $f \in C^{2k}(\mathbf{R})$ of compact support a straightforward integration by parts argument shows

$$f(x) = \frac{1}{2 \cdot (2k-1)!} (|\cdot|^{2k-1} * f^{(2k)})(x).$$

That this also holds for the function ϕ , which grows at infinity, follows from the following more direct argument.

Let $g(x, u)$ be the indefinite integral

$$\int \frac{u^{2k-1}}{((x-u)^2 + c^2)^{(2k+1)/2}} \, du = \frac{p(x, u)}{((x-u)^2 + c^2)^{(2k-1)/2}} + C,$$

discussed in Lemma 6, with C chosen to be 0. Then

$$\int_{-\infty}^{\infty} \frac{|u|^{2k-1}}{((x-u)^2 + c^2)^{(2k+1)/2}} \, du = \lim_{u \rightarrow \infty} g(x, u) + \lim_{u \rightarrow -\infty} g(x, u) - 2g(x, 0).$$

But from the previous lemma the first two terms on the right above cancel and the last term equals

$$\frac{(x^2 + c^2)^{(2k-1)/2}}{kA(k)c^{2k}},$$

which establishes (3.7).

The second part of the lemma is already contained in (2.15) and Lemma 2. ■

4. POLYNOMIALS AS SEMI-INFINITE SUMS OF ψ -SPLINES

Fundamental to the work of Beatson and Powell [2] is that linear polynomials are not only in the space of all bi-infinite combinations of $\psi_{j,2}$'s but also in the space of semi-infinite combinations (modulo a few *edge* ϕ 's). In this section we will obtain an analogous result for ψ -splines of general order.

The proof used in [2] was a direct integration. An alternative collapsing sum argument is as follows. Let

$$\beta(x) = \sum_{j=0}^{\infty} \psi_{j,2k}(x).$$

Then

$$\begin{aligned} \beta(x) &= \lim_{m \rightarrow \infty} \sum_{j=0}^m \psi_{j,2k}(x) \\ &= \frac{1}{2} \lim_{m \rightarrow \infty} \sum_{j=0}^m \{ [t_{j+1}, \dots, t_{j+2k}] \phi(x-u; 2k) \\ &\quad - [t_j, \dots, t_{j+2k-1}] \phi(x-u; 2k) \} \\ &= \frac{1}{2} \{ \lim_{m \rightarrow \infty} [t_{m+1}, \dots, t_{m+2k}] \phi(x-u; 2k) - [t_0, \dots, t_{2k-1}] \phi(x-u; 2k) \} \end{aligned}$$

where all the divided differences are with respect to the u variable. Using the asymptotic expression for $\phi^{(2k-1)}$ (2.7) to express the first term on the right, and the familiar formula

$$[t_l, t_{l+1}, \dots, t_{l+n}] f = \sum_{j=l}^{l+n} \frac{f(t_j)}{\prod_{i=l, i \neq j}^{l+n} (t_j - t_i)}$$

for a divided difference to express the last, it follows that

$$\beta(x) = \frac{1}{2} - \frac{1}{2} \sum_{j=0}^{2k-1} \left[\prod_{i=0, i \neq j}^{2k-1} (t_j - t_i) \right]^{-1} \phi_{j,2k}(x).$$

More generally we have

THEOREM 8 (Polynomials as semi-finite sums of ψ -splines). *Suppose \mathbf{t} satisfies the conditions of Lemma 2 and $\mathbf{d} \in C(2k, \mathbf{t})$. Further suppose $p = \sum_{j=-\infty}^{\infty} d_j \psi_{j,2k}$ is in π_{2k-1} . Then $q = \sum_{j=-\infty}^{\infty} d_j N_{j,2k}$ is also in π_{2k-1} . Furthermore the function s , defined by the semi-infinite sum*

$$s(x) = \sum_{j=0}^{\infty} d_j \psi_{j,2k}(x),$$

can be rewritten as

$$s(x) = \frac{p(x)}{2} + \frac{1}{2} \sum_{l=0}^{2k-1} \lambda_l \phi_{l, 2k}(x),$$

where the vector λ is the unique solution of

$$\sum_{l=0}^{2k-1} \lambda_l (\cdot - t_l)^{2k-1} = q.$$

The theorem also holds in the polynomial spline case $c=0$.

Proof. We consider firstly the case when $c=0$ so that $\psi_{j, 2k}$ is the polynomial B -spline $N_{j, 2k}$ and p and q are identical. Then

$$s(x) = \sum_{j=0}^{\infty} d_j N_{j, 2k}(x)$$

and by the properties of B -splines

$$s(x) = \begin{cases} 0, & x \leq t_0, \\ q(x), & x \geq t_{2k-1}, \end{cases}$$

and has possible jump discontinuities in its $(2k-1)^{st}$ derivative at $t_0, t_1, \dots, t_{2k-1}$. We note that

$$x_+^{2k-1} = \frac{x^{2k-1} + |x|^{2k-1}}{2}$$

has a jump discontinuity of magnitude $(2k-1)!$ in its $(2k-1)^{st}$ derivative at $x=0$ and none elsewhere. Hence,

$$s(x) = \sum_{l=0}^{2k-1} \frac{\lambda_l}{2} \{ (x - t_l)^{2k-1} + |x - t_l|^{2k-1} \},$$

for some constants $\lambda_0, \dots, \lambda_{2k-1}$. This can be rewritten as

$$s(x) = \left\{ \sum_{l=0}^{2k-1} \frac{\lambda_l}{2} (x - t_l)^{2k-1} \right\} + \left\{ \sum_{l=0}^{2k-1} \frac{\lambda_l}{2} |x - t_l|^{2k-1} \right\}.$$

But for $x > t_{2k-1}$ the terms in curly brackets are equal and sum to $q(x)$. Hence the first term equals $q(x)/2$ for all $x > t_{2k-1}$, and since it is a polynomial the equality holds for all $x \in \mathbf{R}$. Thus

$$s(x) = \frac{q(x)}{2} + \left\{ \sum_{l=0}^{2k-1} \frac{\lambda_l}{2} |x - t_l|^{2k-1} \right\}.$$

Comparing these last two expressions for $s(x)$ we find

$$\sum_{l=0}^{2k-1} \lambda_l (\cdot - t_l)^{2k-1} = q.$$

Since $\{(\cdot - t_l)^{2k-1} : l=0, \dots, 2k-1\}$ forms a basis for π_{2k-1} it follows that the coefficients $\lambda_0, \dots, \lambda_{2k-1}$ are uniquely determined by this last equation. This establishes the theorem when $c=0$.

We now turn to the case $c>0$. From Corollary 5, Lemma 1 and Lemma 2

$$q(x) = \sum_{j=-\infty}^{\infty} d_j N_{j, 2k}(x) \quad (4.1)$$

is the unique polynomial in π_{2k-1} such that

$$p = \sum_{j=-\infty}^{\infty} d_j \psi_{j, 2k} = kA(k)c^{2k} \left\{ \sum_{j=-\infty}^{\infty} d_j N_{j, 2k} * (\cdot + c^2)^{-(2k+1)/2} \right\}, \quad (4.2)$$

with the last equality holding term by term. Hence

$$\begin{aligned} s &= \sum_{j=0}^{\infty} d_j \psi_{j, 2k} \\ &= kA(k)c^{2k} \sum_{j=0}^{\infty} d_j \left\{ N_{j, 2k} * (\cdot + c^2)^{-(2k+1)/2} \right\} \\ &= kA(k)c^{2k} \left\{ \frac{q(\cdot)}{2} + \frac{1}{2} \sum_{l=0}^{2k-1} \lambda_l |\cdot - t_l|^{2k-1} \right\} * (\cdot + c^2)^{-(2k+1)/2} \\ &= \frac{p(\cdot)}{2} + \frac{1}{2} \sum_{l=0}^{2k-1} \lambda_l \phi_{l, 2k}(\cdot), \end{aligned}$$

where in the second to last equality we have used the already proven result for $c=0$. The last equality follows from (4.1), (4.2) and Lemma 7. ■

Note that in the special case

$$q = (\cdot - t_0)^{2k-1} = \sum_{j=-\infty}^{\infty} d_j N_{j, 2k},$$

Theorem 8 gives the especially simple expression

$$\sum_{j=0}^{\infty} d_j N_{j, 2k} = \frac{1}{2} \{ (\cdot - t_0)^{2k-1} + |\cdot - t_0|^{2k-1} \} = (\cdot - t_0)_+^{2k-1}.$$

5. APPROXIMATION BY ψ -SPLINES

In this section we consider approximation properties of ψ -splines. We use quasi-interpolants to show Jackson-type error estimates for non-uniform meshes and continuous or continuously differentiable functions. The results are generalisations of some of the results of Buhmann [3, 4] for bi-infinite uniform meshes, and of results of Beatson and Powell [2] for quasi-interpolation on a finite mesh with ordinary multiquadrics.

THEOREM 9. *Let $k \geq 1$, $c > 0$ and mesh $\mathbf{t} : \dots < t_{j-1} < t_j < t_{j+1} < \dots$, with $t_{\pm j} \rightarrow \pm\infty$ as $j \rightarrow \infty$ be given. Suppose the mesh size $h = \sup_j (t_{j+1} - t_j)$ is finite. Then for each function f , uniformly continuous on \mathbf{R} , the quasi-interpolant $\mathcal{L}_{\mathcal{B}} f = \sum_{j=-\infty}^{\infty} f(t_j^*) \psi_{j, 2k}$ satisfies*

$$\|f - \mathcal{L}_{\mathcal{B}} f\|_{L^\infty(\mathbf{R})} \leq \left(k + 1 + \frac{c}{h}\right) \omega(f, h).$$

The same result holds when t_j^ is replaced by t_{j+k} in the definition of $\mathcal{L}_{\mathcal{B}}$.*

Proof. Firstly note that

$$|f(x)| \leq |f(0)| + (1 + |x|) \omega(f, 1), \quad x \in \mathbf{R}.$$

Hence, $|f(x)|$ grows at most linearly as $x \rightarrow \pm\infty$, and $\mathcal{L}_{\mathcal{B}} f$ is well defined by Lemma 2.

From the partition of unity property of the $\psi_{j, 2k}$'s

$$f(x) - (\mathcal{L}_{\mathcal{B}} f)(x) = \sum_{j=-\infty}^{\infty} \{f(x) - f(t_j^*)\} \psi_{j, 2k}(x)$$

From the properties of the modulus of continuity

$$|f(x) - f(t_j^*)| \leq \omega(f, |x - t_j^*|) \leq \left(1 + \frac{|x - t_j^*|}{h}\right) \omega(f, h).$$

Hence using also Lemma 2

$$\begin{aligned} & \left| f(x) - \sum_{j=-\infty}^{\infty} f(t_j^*) \psi_{j, 2k}(x) \right| \\ & \leq kA(k) c^{2k} \int_{-\infty}^{\infty} \frac{\sum_{j=-\infty}^{\infty} |f(x) - f(t_j^*)| N_{j, 2k}(u)}{((x-u)^2 + c^2)^{(2k+1)/2}} du \\ & \leq kA(k) c^{2k} \omega(f, h) \int_{-\infty}^{\infty} \frac{\sum_{j=-\infty}^{\infty} \left(1 + \frac{|x - t_j^*|}{h}\right) N_{j, 2k}(u)}{((x-u)^2 + c^2)^{(2k+1)/2}} du. \end{aligned} \quad (5.1)$$

Now recall that

$$t_j^* = \frac{t_{j+1} + \cdots + t_{j+2k-1}}{2k-1},$$

so that t_j^* is increasing in each of $t_{j+1}, \dots, t_{j+2k-1}$. Hence $\max\{t_j^* - t_j, t_{j+2k} - t_j^*\}$ occurs when all the points are as far apart as possible and is kh . Since $\text{supp}(N_{j, 2k}) = [t_j, t_{j+2k}]$ it follows that $N_{j, 2k}(u)$ is non-zero only when $|u - t_j^*| \leq kh$. When this is the case $|x - t_j^*| \leq |x - u| + kh$. Substituting into (5.1)

$$|f(x) - (\mathcal{L}_{\mathcal{B}} f)(x)| \leq kA(k)c^{2k} \omega(f, h) \int_{-\infty}^{\infty} \frac{(k+1) + \frac{|x-u|}{h}}{((x-u)^2 + c^2)^{(2k+1)/2}} du.$$

Using the values for the integrals given in (2.9) and (2.12) we find

$$|f(x) - (\mathcal{L}_{\mathcal{B}} f)(x)| \leq \left(k + 1 + \frac{c}{h}\right) \omega(f, h).$$

The argument when we replace t_j^* by t_{j+k} is almost identical. ■

COROLLARY 10. *Let $k \in \mathbb{N}$ and mesh $\mathbf{t} : t_0 < t_1 < \cdots < t_n$ be given. Let*

$$\mathcal{B} = \text{span}\{1, \phi_{0, 2k}, \phi_{1, 2k}, \dots, \phi_{n, 2k}\}.$$

Then

$$\text{dist}(f, \mathcal{B}; L^\infty[t_0, t_n]) \leq \left(2k + \frac{c}{h}\right) \omega(f, h)$$

for all $f \in C[t_0, t_n]$, where $h = \max_{0 \leq j \leq n-1} (t_{j+1} - t_j)$ is the mesh size.

Proof. The proof is divided into two cases.

Case 1. $n \leq 2k$. In this case approximate f by the constant function $s(x) = f(t_{\lfloor n/2 \rfloor})$ and note

$$\|f - s\|_{L^\infty[t_0, t_n]} \leq (n - \lfloor n/2 \rfloor) \omega(f, h) \leq 2k\omega(f, h).$$

Case 2. $n > 2k$. In this case extend the mesh to $\pm \infty$ by requiring $t_{j+1} - t_j = h$ for all $j \in \mathbf{Z} \setminus [0, n)$. Then set

$$g(x) = \begin{cases} f(t_{k-1}), & x \leq t_{k-1}, \\ f(x), & t_{k-1} \leq x \leq t_{n-k+1}, \\ f(t_{n-k+1}), & t_{n-k+1} \leq x. \end{cases}$$

Note that $\max\{t_{k-1} - t_0, t_n - t_{n-k+1}\} \leq (k-1)h$ implying that $\|f - g\|_{L^\infty[t_0, t_n]} \leq (k-1)\omega(f, h)$, and also that g is uniformly continuous on \mathbf{R} with $\omega(g, h) \leq \omega(f, h)$. Hence by Theorem 9

$$\left\| g - \sum_{j=-\infty}^{\infty} g(t_{j+k}) \psi_{j, 2k} \right\|_{L^\infty(\mathbf{R})} \leq \left(k + 1 + \frac{c}{h} \right) \omega(f, h).$$

Thus

$$\left\| f - \sum_{j=-\infty}^{\infty} g(t_{j+k}) \psi_{j, 2k} \right\|_{L^\infty[t_0, t_n]} \leq \left(2k + \frac{c}{h} \right) \omega(f, h). \quad (5.2)$$

Then by Lemma 3 and Theorem 8

$$\sum_{j=-\infty}^{-1} g(t_{j+k}) \psi_{j, 2k} = f(t_{k-1}) \sum_{j=-\infty}^{-1} \psi_{j, 2k}$$

is in $\text{span}\{1, \phi_{0, 2k}, \phi_{1, 2k}, \dots, \phi_{2k-1, 2k}\}$ and hence is in the space \mathcal{B} . Similarly

$$\sum_{j=n-2k+1}^{\infty} g(t_{j+k}) \psi_{j, 2k} = f(t_{n-k+1}) \sum_{j=n-2k+1}^{\infty} \psi_{j, 2k}$$

also belongs to the space \mathcal{B} . Since $\psi_{j, 2k} \in \mathcal{B}$ for $j = 0, \dots, n-2k$ the corollary follows from (5.2). ■

THEOREM 11. *Let $k \in \mathbf{N}$ and $k \geq 2$. There exists a constant M , depending only on k , with the following property. Let $c > 0$ and mesh $\mathbf{t} : \dots < t_{j-1} < t_j < t_{j+1} < \dots$ with $t_{\pm j} \rightarrow \pm \infty$ as $j \rightarrow \infty$ be given. Suppose the mesh size $h = \sup_j (t_{j+1} - t_j)$ is finite. Then for each function f with f' uniformly continuous on \mathbf{R} , the quasi-interpolant,*

$$\mathcal{L}_{\mathcal{B}} f = \sum_{j=-\infty}^{\infty} f(t_j^*) \psi_{j, 2k},$$

satisfies

$$\|f - \mathcal{L}_{\mathcal{B}} f\|_{L^\infty(\mathbf{R})} \leq M \left\{ \frac{c^2}{h} + c + h \right\} \omega(f', h).$$

Proof. Firstly note that

$$|f'(x)| \leq |f'(0)| + (1 + |x|) \omega(f', 1) \quad x \in \mathbf{R},$$

so that $|f(x)|$ grows at most quadratically as $x \rightarrow \pm \infty$. Hence $\mathcal{L}_{\mathcal{B}} f$ is well defined by Lemma 2.

Now from Lemma 3, $\mathcal{L}_{\mathcal{B}}$ reproduces linears. It is after all the analogue of the variation diminishing spline. Hence if p is the linear Taylor polynomial of f at x

$$\sum_{j=-\infty}^{\infty} p(t_j^*) \psi_{j, 2k}(x) = p(x) = f(x).$$

Thus the approximation error is

$$|f(x) - (\mathcal{L}_{\mathcal{B}} f)(x)| = \left| \sum_{j=-\infty}^{\infty} \{p(t_j^*) - f(t_j^*)\} \psi_{j, 2k}(x) \right|.$$

Using the bound

$$|f(t_j^*) - p(t_j^*)| \leq |t_j^* - x| \omega(f', |t_j^* - x|)$$

from Taylor's theorem the approximation error is bounded above by

$$\begin{aligned} & \sum_{j=-\infty}^{\infty} |x - t_j^*| \omega(f', |x - t_j^*|) \psi_{j, 2k}(x) \\ & \leq \sum_{j=-\infty}^{\infty} |x - t_j^*| \left(1 + \frac{|x - t_j^*|}{h} \right) \omega(f', h) \psi_{j, 2k}(x). \end{aligned} \quad (5.3)$$

Writing

$$S_{2k}(u) = kA(k) c^{2k} (u^2 + c^2)^{-(2k+1)/2},$$

as before, the right hand side of (5.3) becomes

$$\omega(f', h) \int_{s=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \left\{ \frac{(x - t_j^*)^2}{h} + |x - t_j^*| \right\} N_{j, 2k}(u) S_{2k}(x - u) du. \quad (5.4)$$

But if $N_{j, 2k}(u)$ is non-zero then $|u - t_j^*| \leq kh$ implying

$$|x - t_j^*| \leq |x - u| + kh.$$

Hence (5.4) is bounded above by

$$\omega(f', h) \int_{u=-\infty}^{\infty} \left\{ \frac{1}{h} (x-u)^2 + (2k+1) |x-u| + k(k+1)h \right\} S_{2k}(x-u) du.$$

Since $k \geq 2$, (2.9), (2.12) and (2.21) show there exists a constant M_1 , depending only on k such that

$$\int_{-\infty}^{\infty} |u|^j S_{2k}(u) du \leq M_1 c^j, \quad 0 \leq j \leq 2,$$

and the result follows. ■

COROLLARY 12. *Let $k \in \mathbf{N}$ and $k \geq 2$. There exists a constant M depending only on k with the following property. Let $c > 0$ and mesh $\mathbf{t} : t_0 < t_1 < \dots < t_n$ be given. Let*

$$\mathcal{C} = \text{span}\{1, x, \phi_{0,2k}, \phi_{1,2k}, \dots, \phi_{n,2k}\}.$$

Then

$$\text{dist}(f, \mathcal{C}; L^\infty[t_0, t_n]) \leq M \left\{ \frac{c^2}{h} + c + h \right\} \omega(f', h)$$

for all $f \in C^1[t_0, t_n]$ where $h = \max_{0 \leq j < n-1} (t_{j+1} - t_j)$ is the mesh size.

Proof. The proof is divided into two cases.

Case 1. $n \leq 2k$. In this case approximate f by the linear function $s(x) = f(t_{[n/2]}) + f'(t_{[n/2]})(x - t_{[n/2]})$ and note

$$\|f - s\|_{L^\infty[t_0, t_n]} \leq (n - [n/2]) h \omega(f', kh) \leq k^2 h \omega(f', h).$$

Case 2. $n > 2k$. In this case extend the mesh to $\pm \infty$ by requiring $t_{j+1} - t_j = h$ for all $j \in \mathbf{Z} \setminus [0, n)$. Then set

$$g(x) = \begin{cases} f(t_{-1}^*) + f'(t_{-1}^*)(x - t_{-1}^*), & x \leq t_{-1}^*, \\ f(x), & t_{-1}^* \leq x \leq t_{n-2k+1}^*, \\ f(t_{n-2k+1}^*) + f'(t_{n-2k+1}^*)(x - t_{n-2k+1}^*), & t_{n-2k+1}^* \leq x. \end{cases}$$

Then $\|f - g\|_{L^\infty[t_0, t_n]} \leq (k-1) h \omega(f', (k-1)h) \leq (k-1)^2 h \omega(f', h)$ and g' is uniformly continuous on \mathbf{R} with $\omega(g', h) \leq \omega(f', h)$. By an argument analogous to that in the latter part of the proof of Corollary 10, excepting

that the application of Theorem 9 is replaced by an application of Theorem 11, we find that $(\mathcal{L}_{\mathcal{C}} g) := \sum_{j=-\infty}^{\infty} g(t_j^*) \psi_{j,2k} \in \mathcal{C}$, and

$$\|f - \mathcal{L}_{\mathcal{C}} g\|_{L^\infty[t_0, t_n]} \leq M_2 \left\{ \frac{c^2}{h} + c + h \right\} \omega(f', h). \quad \blacksquare$$

We now turn to the case $k=1$ discussed in Beatson and Powell [2]. Extend $f \in C^1[t_0, t_n]$ outside $[t_0, t_n]$ by appending first degree Taylor polynomials at t_0 and t_n . They show that the operator $\mathcal{L}_B f$ of Theorem 9 applied to this extended f becomes (in the notation of the current paper)

$$\begin{aligned} (\mathcal{L}_B f)(x) &= \sum_{j=-\infty}^{\infty} f(t_j^*) \psi_{j,2}(x) = \sum_{j=-\infty}^{\infty} f(t_{j+1}) \psi_{j,2}(x) \\ &= \frac{f'(t_0)}{2} [(x - t_0) - \phi_0(x)] + \frac{f(t_0)}{2} \left[1 + \frac{\phi_1(x) - \phi_0(x)}{t_1 - t_0} \right] \\ &\quad + \sum_{j=1}^{n-1} f(t_j) \psi_{j-1,2}(x) \\ &\quad + \frac{f(t_n)}{2} \left[1 - \frac{\phi_n(x) - \phi_{n-1}(x)}{t_n - t_{n-1}} \right] + \frac{f'(t_n)}{2} [\phi_n(x) - (t_n - x)]. \end{aligned} \quad (5.5)$$

Note that in [2], ψ_j denotes a combination of $\phi_{j-1,2}$, $\phi_{j,2}$, and $\phi_{j+1,2}$ whereas here it denotes a combination of $\phi_{j,2}$, $\phi_{j+1,2}$ and $\phi_{j+2,2}$. They obtain an estimate for $\|f - \mathcal{L}_B f\|$ when f has a Lipschitz derivative. It is natural therefore to seek an estimate in terms of $\omega(f', h)$.

THEOREM 13. *Let $k=1$. There exists a constant M with the following property. Let a mesh $\mathbf{t}: t_0 < t_1 < \dots < t_n$ be given and $(\mathcal{L}_B f)$ be defined by (5.5), then*

$$\|f - \mathcal{L}_B f\|_{L^\infty[t_0, t_n]} \leq M \left\{ c + h + \frac{c^2}{h} + \frac{c^2}{h} \log \left(1 + \left(\frac{t_n - t_0}{c} \right) \right) \right\} \omega(f', h)$$

for all $f \in C^1[t_0, t_n]$ where h is the mesh size.

Proof. This proof is quite intricate but involves no essentially new ideas. It has therefore been omitted. \blacksquare

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